

PULL-BACK OF QUASI-LOG STRUCTURES

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ABSTRACT. One of the main purposes of this paper is to make the theory of quasi-log schemes more flexible and more useful. More precisely, we prove that the pull-back of a quasi-log scheme by a smooth quasi-projective morphism has a natural quasi-log structure after clarifying the definition of quasi-log schemes. We treat some applications to singular Fano varieties. This paper also contains a proof of the simple connectedness of log canonical Fano varieties.

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1. INTRODUCTION

In [A], the notion of quasi-log structures was introduced in order to prove the cone and contraction theorem for (X, Δ) where X is a normal variety and Δ is an effective \mathbb{R} -divisor such that $K_X + \Delta$ is \mathbb{R} -Cartier. Although the theory of quasi-log schemes is very powerful and useful, it may look much harder than the usual X-method for kawamata log terminal pairs. Moreover, the paper [F4] recovers the main theorem of [A] without using the notion of quasi-log structures. So the theory of quasi-log schemes is not yet popular. We note that the

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framework of [F4] is more similar to the theory of algebraic multiplier ideal sheaves than to the traditional X-method. Recently, the author proved that every quasi-projective semi log canonical pair has a natural quasi-log structure in [F7]. The theory of quasi-log schemes seems to be indispensable for the study of semi log canonical pairs. Now the importance of quasi-log structures is increasing. One of the main purposes of this paper is to clarify the definition of quasi-log structures and make the theory of quasi-log schemes more flexible and more useful.

The following theorem is the main theorem of this paper, which is natural but missing in the literature. For the precise statement, see Theorem 3.11 below.

Theorem 1.1 (Pull-back of quasi-log structures). *Let $[X, \omega]$ be a quasi-log scheme and let $h : X' \rightarrow X$ be a smooth quasi-projective morphism. Then $[X', \omega']$, where $\omega' = h^*\omega \otimes \omega_{X'/X}$ with $\omega_{X'/X} = \det \Omega_{X'/X}^1$, has a natural quasi-log structure induced by h .*

Theorem 1.1 does not directly follow from the original definition of quasi-log schemes. We have to construct a quasi-log resolution of $[X', \omega']$ suitably. We make an important remark. We do not know whether Theorem 1.1 holds true or not without assuming that h is quasi-projective. As a useful special case of Theorem 1.1, we have:

Theorem 1.2 (Finite étale covers). *Let $[X, \omega]$ be a quasi-log scheme and let $h : X' \rightarrow X$ be a finite étale morphism. Then $[X', \omega']$, where $\omega' = h^*\omega$, has a natural quasi-log structure induced by h .*

As an easy application of Theorem 1.2 to singular Fano varieties, we obtain:

Corollary 1.3. *Let $[X, \omega]$ be a projective quasi-log canonical pair such that $-\omega$ is ample, that is, $[X, \omega]$ is a quasi-log canonical Fano variety. Then the algebraic fundamental group of X is trivial, equivalently, X has no non-trivial finite étale covers.*

By Corollary 1.3, it is natural to conjecture:

Conjecture 1.4. *Let $[X, \omega]$ be a projective quasi-log canonical pair such that $-\omega$ is ample. Then X is simply connected.*

In general, there exists an irreducible projective variety whose algebraic fundamental group is trivial and whose topological fundamental group is non-trivial (see Example 6.4). As a special case of Conjecture 1.4, we have:

Conjecture 1.5. *Let (X, Δ) be a projective semi log canonical pair such that $-(K_X + \Delta)$ is ample, that is, (X, Δ) is a semi log canonical Fano variety. Then X is simply connected.*

For the details of semi log canonical pairs, see [F7]. Note that every quasi-projective semi log canonical pair has a natural quasi-log structure with only quasi-log canonical singularities. It is the main theorem of [F7].

It is well known that Conjecture 1.5 holds when (X, Δ) is kawamata log terminal (see [T]). Kento Fujita pointed out that Conjecture 1.5 holds true when (X, Δ) is log canonical (see Theorem 7.1 below). We give Fujita's proof in Section 7 for the reader's convenience.

We summarize the contents of this paper. Section 2 collects some basic definitions and results. In Section 3, we recall the definition of quasi-log schemes and state the main theorem of this paper precisely (see Theorem 3.11). Section 4 is the main part of this paper. Here we discuss the basic properties of quasi-log schemes. The author believes that Section 4 makes the theory of quasi-log schemes more flexible and more useful than Ambro's original framework in [A]. Section 5 is devoted to the proof of the main theorem (see Theorem 1.1 and Theorem 3.11). In Section 6, we treat some applications of the main theorem to singular Fano varieties. We prove that the algebraic fundamental group of a quasi-log canonical Fano variety is always trivial (see Corollary 1.3). In Section 7, we prove that a log canonical Fano variety is simply connected (see Theorem 7.1). The proof of Theorem 7.1 is independent of the theory of quasi-log schemes. Section 8 is an appendix, where we discuss Ambro's original definition of quasi-log schemes.

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We will work over \mathbb{C} , the complex number field, throughout this paper. For the standard notation of the log minimal model program, see, for example, [F4]. For the basic properties and results of semi log canonical pairs, see [F7].

2. PRELIMINARIES

In this section, we collect some basic results and definitions.

Notation 2.1. A pair $[X, \omega]$ consists of a scheme X and an \mathbb{R} -Cartier \mathbb{R} -divisor (or \mathbb{R} -line bundle) ω on X . In this paper, a scheme means a separated scheme of finite type over $\text{Spec } \mathbb{C}$.

Notation 2.2 (Divisors). Let B_1 and B_2 be two \mathbb{R} -Cartier \mathbb{R} -divisors on a scheme X . Then B_1 is linearly (resp. \mathbb{Q} -linearly, or \mathbb{R} -linearly)

equivalent to B_2 , denoted by $B_1 \sim B_2$ (resp. $B_1 \sim_{\mathbb{Q}} B_2$, or $B_1 \sim_{\mathbb{R}} B_2$) if

$$B_1 = B_2 + \sum_{i=1}^k r_i(f_i)$$

such that $f_i \in \Gamma(X, \mathcal{K}_X^*)$ and $r_i \in \mathbb{Z}$ (resp. $r_i \in \mathbb{Q}$, or $r_i \in \mathbb{R}$) for every i . Here, \mathcal{K}_X is the sheaf of total quotient rings of \mathcal{O}_X and \mathcal{K}_X^* is the sheaf of invertible elements in the sheaf of rings \mathcal{K}_X . We note that (f_i) is a *principal Cartier divisor* associated to f_i , that is, the image of f_i by $\Gamma(X, \mathcal{K}_X^*) \rightarrow \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$, where \mathcal{O}_X^* is the sheaf of invertible elements in \mathcal{O}_X .

Let D be a \mathbb{Q} -divisor (resp. an \mathbb{R} -divisor) on an equi-dimensional variety X , that is, D is a finite formal \mathbb{Q} -linear (resp. \mathbb{R} -linear) combination

$$D = \sum_i d_i D_i$$

of irreducible reduced subschemes D_i of codimension one. We define the *round-up* $\lceil D \rceil = \sum_i \lceil d_i \rceil D_i$ (resp. *round-down* $\lfloor D \rfloor = \sum_i \lfloor d_i \rfloor D_i$), where every real number x , $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined by $x \leq \lceil x \rceil < x + 1$ (resp. $x - 1 < \lfloor x \rfloor \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \lfloor D \rfloor$. We put

$$D^{<1} = \sum_{d_i < 1} d_i D_i, \quad D^{\leq 1} = \sum_{d_i \leq 1} d_i D_i, \quad \text{and} \quad D^{=1} = \sum_{d_i = 1} D_i.$$

We can define $D^{\geq 1}$, $D^{>1}$, and so on, analogously. We call D a *boundary* (resp. *subboundary*) \mathbb{R} -divisor if $0 \leq d_i \leq 1$ (resp. $d_i \leq 1$) for every i .

Notation 2.3 (Singularities of pairs). Let X be a normal variety and let Δ be an \mathbb{R} -divisor on X such that $K_X + \Delta$ is \mathbb{R} -Cartier. Let $f : Y \rightarrow X$ be a resolution such that $\text{Exc}(f) \cup f_*^{-1}\Delta$, where $\text{Exc}(f)$ is the exceptional locus of f and $f_*^{-1}\Delta$ is the strict transform of Δ on Y , has a simple normal crossing support. We can write

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i.$$

We say that (X, Δ) is *sub log canonical* (*sub lc*, for short) if $a_i \geq -1$ for every i . We usually write $a_i = a(E_i, X, \Delta)$ and call it the *discrepancy coefficient* of E_i with respect to (X, Δ) . It is well known that there exists the largest Zariski open set U of X such that $(U, \Delta|_U)$ is sub log canonical. If there exist a resolution $f : Y \rightarrow X$ and a divisor E on Y such that $a(E, X, \Delta) = -1$ and $f(E) \cap U \neq \emptyset$, then $f(E)$ is called a *log canonical center* (an *lc center*, for short) with respect to (X, Δ) . If

(X, Δ) is sub log canonical and Δ is effective, then (X, Δ) is called *log canonical* (*lc*, for short).

We note that we can define $a(E_i, X, \Delta)$ in more general settings (see [K2, Definition 2.4]).

Let us recall the definition of simple normal crossing pairs.

Definition 2.4 (Simple normal crossing pairs). We say that the pair (X, D) is *simple normal crossing* at a point $a \in X$ if X has a Zariski open neighborhood U of a that can be embedded in a smooth variety Y , where Y has regular system of parameters $(x_1, \dots, x_p, y_1, \dots, y_r)$ at $a = 0$ in which U is defined by a monomial equation

$$x_1 \cdots x_p = 0$$

and

$$D = \sum_{i=1}^r \alpha_i (y_i = 0)|_U, \quad \alpha_i \in \mathbb{R}.$$

We say that (X, D) is a *simple normal crossing pair* if it is simple normal crossing at every point of X . We say that a simple normal crossing pair (X, D) is *embedded* if there exists a closed embedding $\iota : X \rightarrow M$, where M is a smooth variety of $\dim X + 1$. We call M the *ambient space* of (X, D) . If $(X, 0)$ is a simple normal crossing pair, then X is called a *simple normal crossing variety*. If X is a simple normal crossing variety, then X has only Gorenstein singularities. Thus, it has an invertible dualizing sheaf ω_X . Therefore, we can define the *canonical divisor* K_X such that $\omega_X \simeq \mathcal{O}_X(K_X)$. It is a Cartier divisor on X and is well-defined up to linear equivalence.

Let X be a simple normal crossing variety and let $X = \bigcup_{i \in I} X_i$ be the irreducible decomposition of X . A *stratum* of X is an irreducible component of $X_{i_1} \cap \cdots \cap X_{i_k}$ for some $\{i_1, \dots, i_k\} \subset I$.

Let X be a simple normal crossing variety and let D be a Cartier divisor on X . If (X, D) is a simple normal crossing pair and D is reduced, then D is called a *simple normal crossing divisor* on X .

Let (X, D) be a simple normal crossing pair. Let $\nu : X^\nu \rightarrow X$ be the normalization. We define Θ by the formula

$$K_{X^\nu} + \Theta = \nu^*(K_X + D).$$

Then a *stratum* of (X, D) is an irreducible component of X or the ν -image of a log canonical center of (X^ν, Θ) (see Notation 2.3). When $D = 0$, this definition is compatible with the above definition of the strata of X . When D is a boundary \mathbb{R} -divisor, W is a stratum of (X, D) if and only if W is an slc stratum of (X, D) (see [F7, Definition 2.5]). Note that (X, D) is semi log canonical if D is a boundary \mathbb{R} -divisor.

Notation 2.5. $\pi_1(X)$ denotes the topological fundamental group of X .

3. PULL-BACK OF QUASI-LOG STRUCTURES

In this section, we give a precise statement of Theorem 1.1 (see Theorem 3.11). First, let us recall the definition of *globally embedded simple normal crossing pairs* in order to define quasi-log schemes.

Definition 3.1 (Globally embedded simple normal crossing pairs). Let Y be a simple normal crossing divisor on a smooth variety M and let D be an \mathbb{R} -divisor on M such that $\text{Supp}(D + Y)$ is a simple normal crossing divisor on M and that D and Y have no common irreducible components. We put $B_Y = D|_Y$ and consider the pair (Y, B_Y) . We call (Y, B_Y) a *globally embedded simple normal crossing pair* and M the *ambient space* of (Y, B_Y) .

It is obvious that a globally embedded simple normal crossing pair is an embedded simple normal crossing pair in Definition 2.4.

Let us define *quasi-log schemes* (see also Definition 8.2 below).

Definition 3.2 (Quasi-log schemes). A *quasi-log scheme* is a scheme X endowed with an \mathbb{R} -Cartier \mathbb{R} -divisor (or \mathbb{R} -line bundle) ω on X , a proper closed subscheme $X_{-\infty} \subset X$, and a finite collection $\{C\}$ of reduced and irreducible subschemes of X such that there is a proper morphism $f : (Y, B_Y) \rightarrow X$ from a globally embedded simple normal crossing pair satisfying the following properties:

- (1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.
- (2) The natural map $\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil)$ induces an isomorphism

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor),$$

where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$.

- (3) The collection of subvarieties $\{C\}$ coincides with the image of (Y, B_Y) -strata that are not included in $X_{-\infty}$.

We simply write $[X, \omega]$ to denote the above data

$$(X, \omega, f : (Y, B_Y) \rightarrow X)$$

if there is no risk of confusion. Note that a quasi-log scheme X is the union of $\{C\}$ and $X_{-\infty}$. We also note that ω is called the *quasi-log canonical class* of $[X, \omega]$, which is defined up to \mathbb{R} -linear equivalence. A *relative quasi-log scheme* X/S is a quasi-log scheme X endowed with a proper morphism $\pi : X \rightarrow S$.

Remark 3.3. Let $\text{Div}(Y)$ be the group of Cartier divisors on Y and let $\text{Pic}(Y)$ be the Picard group of Y . Let

$$\delta_Y : \text{Div}(Y) \otimes \mathbb{R} \rightarrow \text{Pic}(Y) \otimes \mathbb{R}$$

be the homomorphism induced by $A \mapsto \mathcal{O}_Y(A)$ where A is a Cartier divisor on Y . When ω is an \mathbb{R} -line bundle in Definition 3.2,

$$f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$$

means

$$f^*\omega = \delta_Y(K_Y + B_Y)$$

in $\text{Pic}(Y) \otimes \mathbb{R}$. Even when ω is an \mathbb{R} -line bundle, we use $-\omega$ to denote the inverse of ω in $\text{Pic}(X) \otimes \mathbb{R}$ (see Corollary 1.3 and Conjecture 1.4) if there is no risk of confusion. If ω is an \mathbb{R} -Cartier \mathbb{R} -divisor on X in Theorem 1.1,

$$h^*\omega \otimes \det \Omega_{X'/X}^1$$

means

$$\delta_{X'}(h^*\omega) \otimes \det \Omega_{X'/X}^1$$

in $\text{Pic}(X') \otimes \mathbb{R}$ where $\delta_{X'} : \text{Div}(X') \otimes \mathbb{R} \rightarrow \text{Pic}(X') \otimes \mathbb{R}$.

We give an important remark on Definition 3.2.

Remark 3.4 (Schemes versus varieties). A quasi-log *scheme* in Definition 3.2 is called a quasi-log *variety* in [A] (see also [F2]). However, X is not always reduced when $X_{-\infty} \neq \emptyset$ (see Example 3.5 below). Therefore, we will use the word *quasi-log schemes* in this paper. Note that X is reduced when $X_{-\infty} = \emptyset$ (see Remark 3.10 below).

Example 3.5 ([A, Examples 4.3.4]). Let X be an effective Cartier divisor on a smooth variety M . Assume that Y , the reduced part of X , is non-empty. We put $\omega = (K_M + X)|_X$. Let $X_{-\infty}$ be the union of the non-reduced components of X . We put $K_Y + B_Y = (K_M + X)|_Y$. Let $f : Y \rightarrow X$ be the closed embedding. Then

$$(X, \omega, f : (Y, B_Y) \rightarrow X)$$

is a quasi-log scheme. Note that X has non-reduced irreducible components if $X_{-\infty} \neq \emptyset$. We also note that f is not surjective if $X_{-\infty} \neq \emptyset$.

Notation 3.6. In Definition 3.2, we sometimes simply say that $[X, \omega]$ is a *quasi-log pair*. The subvarieties C are called the *qlc centers* of $[X, \omega]$, $X_{-\infty}$ is called the *non-qlc locus* of $[X, \omega]$, and $f : (Y, B_Y) \rightarrow X$ is called a *quasi-log resolution* of $[X, \omega]$. We sometimes use $\text{Nqlc}(X, \omega)$ to denote $X_{-\infty}$.

For various applications, the notion of *qlc pairs* is very useful.

Definition 3.7 (Qlc pairs). Let $[X, \omega]$ be a quasi-log pair. We say that $[X, \omega]$ has only *quasi-log canonical singularities* (*qlc singularities*, for short) if $X_{-\infty} = \emptyset$. Assume that $[X, \omega]$ is a quasi-log pair with $X_{-\infty} = \emptyset$. Then we simply say that $[X, \omega]$ is a *qlc pair*.

We give some important remarks on the non-qlc locus $X_{-\infty}$.

Remark 3.8. We put $A = \lceil -(B_Y^{\leq 1}) \rceil$ and $N = \lfloor B_Y^{> 1} \rfloor$. Then we obtain the following big commutative diagram.

$$\begin{array}{ccccccc}
0 & \longrightarrow & f_* \mathcal{O}_Y(A - N) & \longrightarrow & f_* \mathcal{O}_Y(A) & \longrightarrow & f_* \mathcal{O}_N(A) \\
& & \alpha_1 \uparrow & & \alpha_2 \uparrow & & \alpha_3 \uparrow \\
0 & \longrightarrow & f_* \mathcal{O}_Y(-N) & \longrightarrow & f_* \mathcal{O}_Y & \longrightarrow & f_* \mathcal{O}_N \\
& & \beta_1 \uparrow & & \beta_2 \uparrow & & \beta_3 \uparrow \\
0 & \longrightarrow & \mathcal{I}_{X_{-\infty}} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X_{-\infty}} \longrightarrow 0
\end{array}$$

Note that α_i is a natural injection for every i . By an easy diagram chasing,

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_* \mathcal{O}_Y(A - N)$$

factors through $f_* \mathcal{O}_Y(-N)$. Then we obtain β_1 and β_3 . Since α_1 is injective and $\alpha_1 \circ \beta_1$ is an isomorphism, α_1 and β_1 are isomorphisms. Therefore, we obtain that $f(Y) \cap X_{-\infty} = f(N)$. Note that f is not always surjective when $X_{-\infty} \neq \emptyset$. It sometimes happens that $X_{-\infty}$ contains some irreducible components of X . See, for example, Example 3.5.

Remark 3.9 (Semi-normality). By restricting the isomorphism

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_* \mathcal{O}_Y(A - N)$$

to the open subset $U = X \setminus X_{-\infty}$, we obtain

$$\mathcal{O}_U \xrightarrow{\simeq} f_* \mathcal{O}_{f^{-1}(U)}(A).$$

This implies that

$$\mathcal{O}_U \xrightarrow{\simeq} f_* \mathcal{O}_{f^{-1}(U)}$$

because A is effective. Therefore, $f : f^{-1}(U) \rightarrow U$ is surjective and has connected fibers. Note that $f^{-1}(U)$ is a simple normal crossing variety. Thus, U is semi-normal. In particular, $U = X \setminus X_{-\infty}$ is reduced.

Remark 3.10. If the pair $[X, \omega]$ has only qlc singularities, equivalently, $X_{-\infty} = \emptyset$, then X is reduced and semi-normal by Remark 3.9. Note that $f(Y) \cap X_{-\infty} = \emptyset$ if and only if $B_Y = B_Y^{\leq 1}$, equivalently, $B_Y^{> 1} = 0$, by the descriptions in Remark 3.8.

Let us state the main theorem of this paper precisely.

Theorem 3.11 (Main theorem). *Let $[X, \omega]$ be a quasi-log pair as in Definition 3.2. Let X' be a scheme and let $h : X' \rightarrow X$ be a smooth quasi-projective morphism. Then $[X', \omega']$, where $\omega' = h^*\omega \otimes \omega_{X'/X}$ with $\omega_{X'/X} = \det \Omega_{X'/X}^1$, has a natural quasi-log structure induced by h . More precisely, we have the following properties:*

- (i) (Non-qlc locus). *There is a proper closed subscheme $X'_{-\infty} \subset X'$.*
- (ii) (Quasi-log resolution). *There exists a proper morphism $f' : (Y', B_{Y'}) \rightarrow X'$ from a globally embedded simple normal crossing pair $(Y', B_{Y'})$ such that*

$$f'^*\omega' \sim_{\mathbb{R}} K_{Y'} + B_{Y'},$$

and the natural map

$$\mathcal{O}_{X'} \rightarrow f'_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1}) \rceil)$$

induces an isomorphism

$$\mathcal{I}_{X'_{-\infty}} \xrightarrow{\cong} f'_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1}) \rceil - \lfloor B_{Y'}^{>1} \rfloor)$$

where $\mathcal{I}_{X'_{-\infty}}$ is the defining ideal sheaf of $X'_{-\infty}$ and

$$\mathcal{I}_{X'_{-\infty}} = h^*\mathcal{I}_{X_{-\infty}}.$$

- (iii) (Qlc centers). *There is a finite collection $\{C'\}$ of reduced and irreducible subschemes of X' such that $\{C'\} = \{f'^{-1}(C)\}$ and that the collection of subvarieties $\{C'\}$ coincides with the images of $(Y', B_{Y'})$ -strata that are not included in $X'_{-\infty}$.*

Remark 3.12. For the definition and basic properties of *quasi-projective morphisms*, see [G, Chapitre II §5.3. Morphismes quasi-projectifs].

We will prove Theorem 3.11 in Section 5 after we prepare various useful lemmas in Section 4.

We recommend the reader to see [F3] for some basic applications of the theory of quasi-log schemes. The adjunction and vanishing theorem (see, for example, [F3, Theorem 3.6]) is a key result for qlc pairs.

4. ON QUASI-LOG STRUCTURES

The following proposition makes the theory of quasi-log schemes more flexible. It is a key result in this paper.

Proposition 4.1 ([F2, Proposition 3.50]). *Let $f : V \rightarrow W$ be a proper birational morphism between smooth varieties and let B_W be an \mathbb{R} -divisor on W such that $\text{Supp } B_W$ is a simple normal crossing divisor*

on W . Assume that

$$K_V + B_V = f^*(K_W + B_W)$$

and that $\text{Supp } B_V$ is a simple normal crossing divisor on V . Then we have

$$f_*\mathcal{O}_V(\lceil -(B_V^{<1}) \rceil - \lfloor B_V^{>1} \rfloor) \simeq \mathcal{O}_W(\lceil -(B_W^{<1}) \rceil - \lfloor B_W^{>1} \rfloor).$$

Furthermore, let S be a simple normal crossing divisor on W such that $S \subset \text{Supp } B_W^{\bar{=1}}$. Let T be the union of the irreducible components of $B_V^{\bar{=1}}$ that are mapped into S by f . Assume that $\text{Supp } f_*^{-1}B_W \cup \text{Exc}(f)$ is a simple normal crossing divisor on V . Then we have

$$f_*\mathcal{O}_T(\lceil -(B_T^{<1}) \rceil - \lfloor B_T^{>1} \rfloor) \simeq \mathcal{O}_S(\lceil -(B_S^{<1}) \rceil - \lfloor B_S^{>1} \rfloor),$$

where $(K_V + B_V)|_T = K_T + B_T$ and $(K_W + B_W)|_S = K_S + B_S$.

Proof. By $K_V + B_V = f^*(K_W + B_W)$, we obtain

$$\begin{aligned} K_V &= f^*(K_W + B_W^{\bar{=1}} + \{B_W\}) \\ &\quad + f^*(\lfloor B_W^{<1} \rfloor + \lfloor B_W^{>1} \rfloor) - (\lfloor B_V^{<1} \rfloor + \lfloor B_V^{>1} \rfloor) - B_V^{\bar{=1}} - \{B_V\}. \end{aligned}$$

If $a(\nu, W, B_W^{\bar{=1}} + \{B_W\}) = -1$ for a prime divisor ν over W , then we can check that $a(\nu, W, B_W) = -1$ by using [KM, Lemma 2.45]. Since

$$f^*(\lfloor B_W^{<1} \rfloor + \lfloor B_W^{>1} \rfloor) - (\lfloor B_V^{<1} \rfloor + \lfloor B_V^{>1} \rfloor)$$

is Cartier, we can easily see that

$$f^*(\lfloor B_W^{<1} \rfloor + \lfloor B_W^{>1} \rfloor) = \lfloor B_V^{<1} \rfloor + \lfloor B_V^{>1} \rfloor + E,$$

where E is an effective f -exceptional divisor. Thus, we obtain

$$f_*\mathcal{O}_V(\lceil -(B_V^{<1}) \rceil - \lfloor B_V^{>1} \rfloor) \simeq \mathcal{O}_W(\lceil -(B_W^{<1}) \rceil - \lfloor B_W^{>1} \rfloor).$$

Next, we consider the short exact sequence:

$$\begin{aligned} 0 &\rightarrow \mathcal{O}_V(\lceil -(B_V^{<1}) \rceil - \lfloor B_V^{>1} \rfloor - T) \\ &\rightarrow \mathcal{O}_V(\lceil -(B_V^{<1}) \rceil - \lfloor B_V^{>1} \rfloor) \rightarrow \mathcal{O}_T(\lceil -(B_T^{<1}) \rceil - \lfloor B_T^{>1} \rfloor) \rightarrow 0. \end{aligned}$$

Since $T = f^*S - F$, where F is an effective f -exceptional divisor, we can easily see that

$$f_*\mathcal{O}_V(\lceil -(B_V^{<1}) \rceil - \lfloor B_V^{>1} \rfloor - T) \simeq \mathcal{O}_W(\lceil -(B_W^{<1}) \rceil - \lfloor B_W^{>1} \rfloor - S).$$

We note that

$$\begin{aligned} (\lceil -(B_V^{<1}) \rceil - \lfloor B_V^{>1} \rfloor - T) - (K_V + \{B_V\} + B_V^{\bar{=1}} - T) \\ = -f^*(K_W + B_W). \end{aligned}$$

Therefore, every associated prime of $R^1 f_*\mathcal{O}_V(\lceil -(B_V^{<1}) \rceil - \lfloor B_V^{>1} \rfloor - T)$ is the generic point of the f -image of some stratum of $(V, \{B_V\} + B_V^{\bar{=1}} - T)$ (see, for example, [F4, Theorem 6.3 (i)]).

Claim. *No strata of $(V, \{B_V\} + B_V^{-1} - T)$ are mapped into S by f .*

Proof of Claim. Assume that there is a stratum C of $(V, \{B_V\} + B_V^{-1} - T)$ such that $f(C) \subset S$. Note that

$$\text{Supp } f^*S \subset \text{Supp } f_*^{-1}B_W \cup \text{Exc}(f)$$

and

$$\text{Supp } B_V^{-1} \subset \text{Supp } f_*^{-1}B_W \cup \text{Exc}(f).$$

Since C is also a stratum of (V, B_V^{-1}) and

$$C \subset \text{Supp } f^*S,$$

there exists an irreducible component G of B_V^{-1} such that

$$C \subset G \subset \text{Supp } f^*S.$$

Therefore, by the definition of T , G is an irreducible component of T because $f(G) \subset S$ and G is an irreducible component of B_V^{-1} . So, C is not a stratum of $(V, \{B_V\} + B_V^{-1} - T)$. It is a contradiction. \square

On the other hand, $f(T) \subset S$. Therefore,

$$f_*\mathcal{O}_T(\lceil -(B_T^{<1}) \rceil - \lfloor B_T^{>1} \rfloor) \rightarrow R^1f_*\mathcal{O}_V(\lceil -(B_Z^{<1}) \rceil - \lfloor B_Z^{>1} \rfloor - T)$$

is a zero map by Claim. Thus, we obtain

$$f_*\mathcal{O}_T(\lceil -(B_T^{<1}) \rceil - \lfloor B_T^{>1} \rfloor) \simeq \mathcal{O}_S(\lceil -(B_S^{<1}) \rceil - \lfloor B_S^{>1} \rfloor)$$

by the following commutative diagram.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathcal{O}_W(\lceil -(B_W^{<1}) \rceil - \lfloor B_W^{>1} \rfloor - S) & \xrightarrow{\simeq} & f_*\mathcal{O}_V(\lceil -(B_V^{<1}) \rceil - \lfloor B_V^{>1} \rfloor - T) \\
\downarrow & & \downarrow \\
\mathcal{O}_W(\lceil -(B_W^{<1}) \rceil - \lfloor B_W^{>1} \rfloor) & \xrightarrow{\simeq} & f_*\mathcal{O}_V(\lceil -(B_V^{<1}) \rceil - \lfloor B_V^{>1} \rfloor) \\
\downarrow & & \downarrow \\
\mathcal{O}_S(\lceil -(B_S^{<1}) \rceil - \lfloor B_S^{>1} \rfloor) & \longrightarrow & f_*\mathcal{O}_T(\lceil -(B_T^{<1}) \rceil - \lfloor B_T^{>1} \rfloor) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

We finish the proof. \square

It is easy to check:

Proposition 4.2. *In Proposition 4.1, let C' be an lc center of (V, B_V) contained in T . Then $f(C')$ is an lc center of (W, B_W) contained in S or $f(C')$ is contained in $\text{Supp } B_W^{>1}$. Let C be an lc center of (W, B_W) contained in S . Then there exists an lc center C' of (V, B_V) contained in T such that $f(C') = C$.*

The following important theorem is missing in [F2].

Theorem 4.3. *In Definition 3.2, we may assume that the ambient space M of the globally embedded simple normal crossing pair (Y, B_Y) is quasi-projective. In particular, Y is quasi-projective.*

Proof. In Definition 3.2, we may assume that $D + Y$ is an \mathbb{R} -divisor on a smooth variety M such that $\text{Supp}(D + Y)$ is a simple normal crossing divisor on M , D and Y have no common irreducible components, and $B_Y = D|_Y$ as in Definition 3.1. Let $g : M' \rightarrow M$ be a projective birational morphism from a smooth quasi-projective variety M' with the following properties:

- (i) $K_{M'} + B_{M'} = g^*(K_M + D + Y)$,
- (ii) $\text{Supp } B_{M'}$ is a simple normal crossing divisor on M' , and
- (iii) $\text{Supp } g_*^{-1}(D + Y) \cup \text{Exc}(g)$ is also a simple normal crossing divisor on M' .

Let Y' be the union of the irreducible components of $B_{M'}^{-1}$ that are mapped into Y by g . We put

$$(K_{M'} + B_{M'})|_{Y'} = K_{Y'} + B_{Y'}.$$

Then

$$g_* \mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1}) \rceil - \lfloor B_{Y'}^{>1} \rfloor) \simeq \mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor)$$

by Proposition 4.1. This implies that

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_* g_* \mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1}) \rceil - \lfloor B_{Y'}^{>1} \rfloor).$$

By the construction,

$$K_{Y'} + B_{Y'} = g^*(K_Y + B_Y) \sim_{\mathbb{R}} g^* f^* \omega.$$

By Proposition 4.2, the collection of subvarieties $\{C\}$ in Definition 3.2 coincides with the image of $(Y', B_{Y'})$ -strata that are not contained in $X_{-\infty}$. Therefore, by replacing M and (Y, B_Y) with M' and $(Y', B_{Y'})$, we may assume that the ambient space M is quasi-projective. \square

Theorem 4.3 makes the theory of quasi-log schemes flexible and useful.

Lemma 4.4. *Let (Y, B_Y) be a simple normal crossing pair. Let V be a smooth variety such that $Y \subset V$. Then we can construct a sequence of blow-ups*

$$V_k \rightarrow V_{k-1} \rightarrow \cdots \rightarrow V_0 = V$$

with the following properties.

- (1) $\sigma_{i+1} : V_{i+1} \rightarrow V_i$ is the blow-up along a smooth irreducible component of $\text{Supp } B_{Y_i}$ for every $i \geq 0$.
- (2) We put $Y_0 = Y$ and $B_{Y_0} = B_Y$. Let Y_{i+1} be the strict transform of Y_i for every $i \geq 0$.
- (3) We define $K_{Y_{i+1}} + B_{Y_{i+1}} = \sigma_{i+1}^*(K_{Y_i} + B_{Y_i})$ for every $i \geq 0$.
- (4) There exists an \mathbb{R} -divisor D on V_k such that $D|_{Y_k} = B_{Y_k}$.
- (5) $\sigma_* \mathcal{O}_{Y_k}(\lceil -(B_{Y_k}^{<1}) \rceil - \lfloor B_{Y_k}^{>1} \rfloor) \simeq \mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor)$, where $\sigma : V_k \rightarrow V_{k-1} \rightarrow \cdots \rightarrow V_0 = V$.

Proof. It is sufficient to check (5). All the other properties are obvious by the construction of the sequence of blow-ups. By an easy calculation of discrepancy coefficients similar to the proof of Proposition 4.1, we can check that

$$\sigma_{i+1*} \mathcal{O}_{V_{i+1}}(\lceil -(B_{Y_{i+1}}^{<1}) \rceil - \lfloor B_{Y_{i+1}}^{>1} \rfloor) \simeq \mathcal{O}_{V_i}(\lceil -(B_{Y_i}^{<1}) \rceil - \lfloor B_{Y_i}^{>1} \rfloor)$$

for every i . This implies the desired isomorphism. \square

We can easily check:

Lemma 4.5. *In Lemma 4.4, let C' be a stratum of (Y_k, B_{Y_k}) . Then $\sigma(C')$ is a stratum of (Y, B_Y) . Let C be a stratum of (Y, B_Y) . Then there is a stratum C' of (Y_k, B_{Y_k}) such that $\sigma(C') = C$.*

The following lemma is easy but very useful (cf. [K2, Proposition 10.59]).

Lemma 4.6. *Let Y be a simple normal crossing variety. Let V be a smooth quasi-projective variety such that $Y \subset V$. Let $\{P_i\}$ be any finite set of closed points of Y . Then we can find a quasi-projective variety W such that $Y \subset W \subset V$, $\dim W = \dim Y + 1$, and W is smooth at P_i for every i .*

Proof. Let \overline{V} be the closure of V in a projective space and let \overline{Y} be the closure of Y in \overline{V} . We take a sufficiently large positive integer d such that the scheme theoretic base locus of $|\mathcal{O}_{\overline{V}}(dH) \otimes \mathcal{I}_{\overline{Y}}|$ is \overline{Y} near P_i for every i , where H is a very ample Cartier divisor on \overline{V} and $\mathcal{I}_{\overline{Y}}$ is the defining ideal sheaf of \overline{Y} on \overline{V} . By taking a complete intersection of $(\dim V - \dim Y - 1)$ general members of $|\mathcal{O}_{\overline{V}}(dH) \otimes \mathcal{I}_{\overline{Y}}|$, we obtain $\overline{W} \supset \overline{Y}$ such that \overline{W} is smooth at P_i for every i . Note that we used the fact that \overline{Y} has only hypersurface singularities near P_i

for every i . We put $W = \overline{W} \cap V$. By the construction, $W \subset V$ and $\dim W = \dim Y + 1$. \square

Of course, we can not always make W smooth in Lemma 4.6.

Example 4.7 ([F2, Example 3.62]). Let $V \subset \mathbb{P}^5$ be the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$. In this case, there are no smooth hypersurfaces of \mathbb{P}^5 containing V . We can check it as follows.

If there exists a smooth hypersurface S such that $V \subset S \subset \mathbb{P}^5$, then $\rho(V) = \rho(S) = \rho(\mathbb{P}^5) = 1$ by the Lefschetz hyperplane theorem. It is a contradiction because $\rho(V) = 2$.

By the above results, we can prove the final lemma in this section.

Lemma 4.8. *Let (Y, B_Y) be a simple normal crossing pair such that Y is quasi-projective. Then there exist a globally embedded simple normal crossing pair (Z, B_Z) and a morphism $\sigma : Z \rightarrow Y$ such that*

$$K_Z + B_Z = \sigma^*(K_Y + B_Y)$$

and

$$\sigma_* \mathcal{O}_Z(\lceil -(B_Z^{<1}) \rceil - \lfloor B_Z^{>1} \rfloor) \simeq \mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor).$$

Moreover, let C' be a stratum of (Z, B_Z) . Then $\sigma(C')$ is a stratum of (Y, B_Y) or $\sigma(C')$ is contained in $\text{Supp } B_Y^{>1}$. Let C be a stratum of (Y, B_Y) . Then there exists a stratum C' of (Z, B_Z) such that $\sigma(C') = C$.

Proof. Let V be a smooth quasi-projective variety such that $Y \subset V$. By Lemma 4.4 and Lemma 4.5, we may assume that there exists an \mathbb{R} -divisor D on V such that $D|_Y = B_Y$. Then we apply Lemma 4.6. We can find a quasi-projective variety W such that $Y \subset W \subset V$, $\dim W = \dim Y + 1$, and W is smooth at the generic point of any stratum of $(Y, \text{Supp } B_Y)$. Of course, we can make $W \not\subset \text{Supp } D$ (see the proof of Lemma 4.6). We apply Hironaka's resolution to W and use Szabó's resolution lemma (see, for example, [F1, 3.5 Resolution lemma]). More precisely, we take blow-ups outside U , where U is the largest Zariski open set of W such that $(Y, B_Y)|_U$ is a globally embedded simple normal crossing pair. Then we obtain a desired globally embedded simple normal crossing pair (Z, B_Z) . More precisely, we can check that (Z, B_Z) has the desired properties by an easy calculation of discrepancy coefficients similar to the proof of Proposition 4.1. \square

Therefore, we obtain the following statement, which is the main result of this section.

Theorem 4.9. *In Definition 3.2, it is sufficient to assume that (Y, B_Y) is a quasi-projective (not necessarily embedded) simple normal crossing pair.*

Proof. We only assume that (Y, B_Y) is a simple normal crossing pair in Definition 3.2. We assume that Y is quasi-projective. Then we apply Lemma 4.8 to (Y, B_Y) . Let $\sigma : (Z, B_Z) \rightarrow (Y, B_Y)$ be as in Lemma 4.8. Then

$$(X, \omega, f \circ \sigma : (Z, B_Z) \rightarrow X)$$

is a quasi-log scheme in the sense of Definition 3.2. \square

Proposition 4.10 shows that it is not so easy to apply Chow's lemma directly to make (Y, B_Y) quasi-projective in Definition 3.2.

Proposition 4.10 ([F2, Proposition 3.65]). *There exists a complete simple normal crossing variety Y with the following property. If $f : Z \rightarrow Y$ is a proper surjective morphism from a simple normal crossing variety Z such that f is an isomorphism at the generic point of any stratum of Z , then Z is non-projective.*

Proof. We take a smooth complete non-projective toric variety X . We put $V = X \times \mathbb{P}^1$. Then V is a toric variety. We consider $Y = V \setminus T$, where T is the big torus of V . We will see that Y has the desired property. By the above construction, there is an irreducible component Y' of Y that is isomorphic to X . Let Z' be the irreducible component of Z mapped onto Y' by f . So, it is sufficient to see that Z' is not projective. On $Y' \simeq X$, there is a torus invariant effective one cycle C such that C is numerically trivial. By the construction and the assumption, $g = f|_{Z'} : Z' \rightarrow Y' \simeq X$ is birational and an isomorphism over the generic point of any torus invariant curve on $Y' \simeq X$. We note that any torus invariant curve on $Y' \simeq X$ is a stratum of Y . We assume that Z' is projective, then there is a very ample effective divisor A on Z' such that A does not contain any irreducible components of the inverse image of C . Then $B = f_*A$ is an effective Cartier divisor on $Y' \simeq X$ such that $\text{Supp } B$ contains no irreducible components of C . It is a contradiction because $\text{Supp } B \cap C \neq \emptyset$ and C is numerically trivial. \square

Proposition 4.10 is the main reason why we proved Theorem 4.3 for the proof of our main theorem: Theorem 1.1 and Theorem 3.11.

5. PROOF OF THE MAIN THEOREM

Now the proof of the main theorem (see Theorem 1.1 and Theorem 3.11) is almost obvious.

Proof of Theorem 3.11. Let $f : (Y, B_Y) \rightarrow X$ be a quasi-log resolution as in Definition 3.2. By Theorem 4.3, we may assume that Y is quasi-projective. We consider the fiber product $Y' = Y \times_X X'$.

$$\begin{array}{ccc} Y' & \xrightarrow{h'} & Y \\ f' \downarrow & & \downarrow f \\ X' & \xrightarrow{h} & X \end{array}$$

We put $B_{Y'} = h'^*B_Y$. Then $(Y', B_{Y'})$ is a quasi-projective simple normal crossing pair because h is a smooth quasi-projective morphism and (Y, B_Y) is a quasi-projective simple normal crossing pair. Since $K_Y + B_Y \sim_{\mathbb{R}} f^*\omega$, we have

$$\begin{aligned} f'^*\omega' &= f'h^*\omega \otimes f'^*\omega_{X'/X} \\ &= h'f^*\omega \otimes \omega_{Y'/Y} \\ &\sim_{\mathbb{R}} h'^*(K_Y + B_Y) \otimes \omega_{Y'/Y} \\ &= K_{Y'} + B_{Y'}. \end{aligned}$$

By the flat base change theorem, we have

$$\begin{aligned} h^*\mathcal{I}_{X-\infty} &= h^*f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor) \\ &\simeq f'_*h'^*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor) \\ &\simeq f'_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1}) \rceil - \lfloor B_{Y'}^{>1} \rfloor). \end{aligned}$$

Finally, by Theorem 4.9, we may assume that $(Y', B_{Y'})$ is a globally embedded simple normal crossing pair. Therefore,

$$(X', \omega', f' : (Y', B_{Y'}) \rightarrow X')$$

gives us the desired quasi-log structure. \square

Theorem 1.2 is a special case of Theorem 1.1 and Theorem 3.11.

Proof of Theorem 1.2. Note that $\Omega_{X'/X}^1 = 0$ because $h : X' \rightarrow X$ is a finite étale morphism. Therefore, we have $\omega' = h^*\omega \otimes \det \Omega_{X'/X}^1 \simeq h^*\omega$. Thus Theorem 1.2 follows from Theorem 1.1 and Theorem 3.11. \square

6. APPLICATIONS TO QUASI-LOG CANONICAL FANO VARIETIES

Let us recall the vanishing theorem for projective qlc pairs. It is a very special case of [F2, Theorem 3.39 (ii)]. For the details of various vanishing theorems for reducible varieties, see [F5] and [F6].

Theorem 6.1 (Vanishing theorem for qlc pairs). *Let $[X, \omega]$ be a projective qlc pair and let L be a Cartier divisor on X such that $L - \omega$ is ample. Then $H^i(X, \mathcal{O}_X(L)) = 0$ for every $i > 0$.*

We give a proof of Theorem 6.1 for the reader's convenience.

Proof. Let $f : (Y, B_Y) \rightarrow X$ be a quasi-log resolution as in Definition 3.2. Then

$$\begin{aligned} f^*(L - \omega) &\sim_{\mathbb{R}} f^*L - (K_Y + B_Y) \\ &= f^*L + \lceil -(B_Y^{\leq 1}) \rceil - (K_Y + \{B_Y\} + B_Y^{-1}) \end{aligned}$$

because $B_Y = B_Y^{\leq 1}$ (see Remark 3.10). Therefore, we have

$$H^i(X, f_*\mathcal{O}_Y(f^*L + \lceil -(B_Y^{\leq 1}) \rceil)) = 0$$

for every $i > 0$ (see, for example, [F5, Theorem 1.1 (ii)]). Note that

$$f_*\mathcal{O}_Y(f^*L + \lceil -(B_Y^{\leq 1}) \rceil) \simeq \mathcal{O}_X(L) \otimes f_*\mathcal{O}_Y(\lceil -(B_Y^{\leq 1}) \rceil) \simeq \mathcal{O}_X(L)$$

because $X_{-\infty} = \emptyset$ (see Remark 3.10). This implies that

$$H^i(X, \mathcal{O}_X(L)) = 0$$

for every $i > 0$. □

By combining Theorem 6.1 with Theorem 3.11, we can easily check Corollary 1.3.

Proof of Corollary 1.3. Without loss of generality, we may assume that X is connected. Since $-\omega$ is ample, $H^i(X, \mathcal{O}_X) = 0$ for every $i > 0$ by Theorem 6.1. Therefore, we have $\chi(X, \mathcal{O}_X) = 1$. Let $f : \tilde{X} \rightarrow X$ be a non-trivial finite étale morphism from a connected scheme \tilde{X} . By Theorem 3.11, the pair $[\tilde{X}, \tilde{\omega}]$, where $\tilde{\omega} = f^*\omega$, is a qlc pair such that $-\tilde{\omega}$ is ample. Thus, $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ for every $i > 0$ by Theorem 6.1 again. This implies $\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 1$. By the Riemann–Roch formula (see, for example, [Ft, Example 18.3.9]), we have

$$\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \deg f \cdot \chi(X, \mathcal{O}_X).$$

Therefore, we obtain $\deg f = 1$. This means that X has no non-trivial finite étale covers, equivalently, the algebraic fundamental group of X is trivial. □

As a direct consequence of Corollary 1.3 and the main theorem of [F7], we have:

Corollary 6.2. *Let (X, Δ) be a projective semi log canonical pair such that $-(K_X + \Delta)$ is ample, that is, (X, Δ) is a semi log canonical Fano variety. Then the algebraic fundamental group of X is trivial.*

Proof. By [F7], $[X, K_X + \Delta]$ has a natural quasi-log structure with only qlc singularities. Therefore, Corollary 6.2 is a special case of Corollary 1.3. □

Note that a union of some slc strata of a semi log canonical Fano variety is a quasi-log canonical Fano variety by Example 6.3.

Example 6.3. Let (X, Δ) be a connected projective semi log canonical pair such that $-(K_X + \Delta)$ is ample. Let W be a union of some slc strata of (X, Δ) with the reduced scheme structure. Then $[W, \omega]$, where $\omega = (K_X + \Delta)|_W$, is a projective qlc pair such that $-\omega$ is ample by adjunction (see [F7, Theorem 1.13]). By [F7, Theorem 1.11], we obtain $H^1(X, \mathcal{I}_W) = 0$ where \mathcal{I}_W is the defining ideal sheaf of W on X . Therefore, we obtain $H^0(W, \mathcal{O}_W) = \mathbb{C}$ by the surjection

$$\mathbb{C} = H^0(X, \mathcal{O}_X) \rightarrow H^0(W, \mathcal{O}_W).$$

This implies that W is connected.

The author learned the following example from Tetsushi Ito.

Example 6.4 (Topological versus algebraic). We consider the Higman group G . It is generated by 4 elements a, b, c, d with the relations

$$a^{-1}ba = b^2, \quad b^{-1}cb = c^2, \quad c^{-1}dc = d^2, \quad d^{-1}ad = a^2.$$

It is well known that G has no non-trivial finite quotients. By [S, Theorem 12.1], there is an irreducible projective variety X such that $\pi_1(X) \simeq G$. In this case, the algebraic fundamental group of X , which is the profinite completion of $\pi_1(X)$, is trivial.

Example 6.4 shows that Conjecture 1.4 does not directly follow from Corollary 1.3.

We give a non-trivial example of reducible semi log canonical Fano varieties.

Example 6.5. We consider the lattice $N = \mathbb{Z}^3$. Let n be an integer with $n \geq 3$. We consider a convex polyhedron P in $N_{\mathbb{R}} = N \otimes \mathbb{R} \simeq \mathbb{R}^3$ whose vertices are $v_0, v_1, \dots, v_n \in N$ such that $v_0 = (0, 0, -1)$ and that the third coordinates of v_1, \dots, v_n are 1. Assume that P contains $(0, 0, 0)$ in its interior. Then the cones spanned by $(0, 0, 0)$ and faces of P subdivide \mathbb{R}^3 into $n + 1$ three-dimensional cones. This subdivision of \mathbb{R}^3 corresponds to a complete toric threefold X . Then we have the following properties.

- (1) $-K_X$ is ample since P is convex.
- (2) $D_0 \sim D_1 + \dots + D_n$ and D_0 is \mathbb{Q} -Cartier, where D_i is the torus invariant prime divisor on X associated to v_i for every i .
- (3) Let $x \in X$ be the torus invariant closed point associated to the cone spanned by v_1, v_2, \dots, v_n . Then $X \setminus x$ is \mathbb{Q} -factorial, but X is not \mathbb{Q} -factorial when $n \geq 4$.

- (4) We put $\Delta = D_1 + \cdots + D_n$. Then (X, Δ) is a log canonical Fano threefold. Note that $-(K_X + \Delta) \sim D_0$.
- (5) We put $W = \lfloor \Delta \rfloor = \Delta$ and

$$K_W + \Delta_W = (K_X + \Delta)|_W.$$

Then (W, Δ_W) is a semi log canonical Fano surface. Note that W is Cohen–Macaulay since W is \mathbb{Q} -Cartier.

This W shows that the number of irreducible components of semi log canonical Fano surfaces is not bounded.

We recommend the reader who can read Japanese to see [F8] for some related topics and open problems on singular Fano varieties.

7. SIMPLE CONNECTEDNESS OF LOG CANONICAL FANO VARIETIES

In this section, we prove that a log canonical Fano variety is always simply connected. Theorem 7.1 is Fujita’s answer to the author’s question.

Theorem 7.1 (Kento Fujita). *Let (X, Δ) be a projective log canonical pair such that $-(K_X + \Delta)$ is ample, that is, (X, Δ) is a log canonical Fano variety. Then X is simply connected.*

Proof. First of all, we may assume that X is connected. Without loss of generality, we may assume that Δ is a \mathbb{Q} -divisor by perturbing Δ slightly. Then, by [HM, Corollary 1.3 (2)], X is rationally chain connected. Since X is normal and rationally chain connected, $\pi_1(X)$ is finite (see, for example, [K1, 4.13 Theorem]). Let $f : \tilde{X} \rightarrow X$ be the universal cover of X . Since $\pi_1(X)$ is finite, f is finite and étale. It is obvious that $(\tilde{X}, \tilde{\Delta})$ is log canonical and $-(K_{\tilde{X}} + \tilde{\Delta})$ is ample, where

$$K_{\tilde{X}} + \tilde{\Delta} = f^*(K_X + \Delta).$$

By [F4, Theorem 8.1], we have

$$H^i(X, \mathcal{O}_X) = H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$$

for every $i > 0$. This implies

$$\chi(X, \mathcal{O}_X) = \chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 1.$$

On the other hand,

$$\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \deg f \cdot \chi(X, \mathcal{O}_X)$$

holds by the Riemann–Roch formula (see, for example, [Ft, Example 18.3.9]). Thus we obtain $\deg f = 1$. Therefore, X is simply connected. \square

Remark 7.2. By [HM, Corollary 1.3 (2)], we can easily see that a semi log canonical Fano variety is rationally chain connected. However, [K1, 4.13 Theorem] does not always hold for *non-normal* rationally chain connected varieties. Note that a nodal rational curve C is rationally chain connected such that $\pi_1(C)$ is infinite. Therefore, the proof of Theorem 7.1 does not work for semi log canonical Fano varieties.

The following well-known example shows some subtleties on log canonical Fano varieties.

Example 7.3. Let $C \subset \mathbb{P}^2$ be a smooth cubic curve and let $X \subset \mathbb{P}^3$ be the cone over $C \subset \mathbb{P}^2$. then X is a Gorenstein log canonical surface such that $-K_X$ is ample. It is easy to see that X is rationally chain connected and that $\pi_1(X) = \{1\}$ by Theorem 7.1. Let $f : Y \rightarrow X$ be the blow-up at P where P is the vertex of X . Then

$$K_Y + E = f^*K_X.$$

The pair (Y, E) is purely log terminal and $-(K_Y + E)$ is big and semi-ample. Note that the exceptional curve E is isomorphic to C and that Y is a \mathbb{P}^1 -bundle over C . Therefore, it is easy to see that Y is not rationally chain connected and $\pi_1(Y) \neq \{1\}$.

Example 7.4 is a non-trivial example of *irreducible non-normal* semi log canonical Fano varieties.

Example 7.4. We put $X = (x^2w - zy^2 = 0) \subset \mathbb{P}^3$. Then X is a Gorenstein Fano variety with only semi log canonical singularities. Note that X is irreducible and non-normal. By using the van Kampen theorem, we see that $\pi_1(X) = \{1\}$.

8. APPENDIX: AMBRO'S ORIGINAL DEFINITION

In this section, we prove that our definition of quasi-log schemes (see Definition 3.2) is equivalent to Ambro's original definition in [A].

First, let us recall the definition of *normal crossing pairs*. We need it for Ambro's original definition of quasi-log schemes in [A].

Definition 8.1 (Normal crossing pairs). A variety X has *normal crossing* singularities if, for every closed point $x \in X$,

$$\widehat{\mathcal{O}}_{X,x} \simeq \frac{\mathbb{C}[[x_0, \dots, x_N]]}{(x_0 \cdots x_k)}$$

for some $0 \leq k \leq N$, where $N = \dim X$. Let X be a normal crossing variety. We say that a reduced divisor D on X is *normal crossing* if,

in the above notation, we have

$$\widehat{\mathcal{O}}_{D,x} \simeq \frac{\mathbb{C}[[x_0, \dots, x_N]]}{(x_0 \cdots x_k, x_{i_1} \cdots x_{i_l})}$$

for some $\{i_1, \dots, i_l\} \subset \{k+1, \dots, N\}$. We say that the pair (X, B) is a *normal crossing pair* if the following conditions are satisfied.

- (1) X is a normal crossing variety, and
- (2) B is an \mathbb{R} -Cartier \mathbb{R} -divisor whose support is normal crossing on X .

We say that a normal crossing pair (X, B) is *embedded* if there exists a closed embedding $\iota : X \rightarrow M$, where M is a smooth variety of dimension $\dim X + 1$. We call M the *ambient space* of (X, B) . We put

$$K_{X^\nu} + \Theta = \nu^*(K_X + B),$$

where $\nu : X^\nu \rightarrow X$ is the normalization of X . A *stratum* of (X, B) is an irreducible component of X or the image of some lc center of (X^ν, Θ) on X .

It is obvious that a *simple normal crossing pair* in Definition 2.4 is a *normal crossing pair* in Definition 8.1. Note that the differences between normal crossing varieties and simple normal crossing varieties sometimes cause some subtle troubles (see, for example, [F1, 3.6 Whitney umbrella]).

Let us recall Ambro's original definition of quasi-log schemes (see [A]).

Definition 8.2 (Quasi-log schemes). A *quasi-log scheme* is a scheme X endowed with an \mathbb{R} -Cartier \mathbb{R} -divisor (or \mathbb{R} -line bundle) ω , a proper closed subscheme $X_{-\infty} \subset X$, and a finite collection $\{C\}$ of reduced and irreducible subschemes of X such that there is a proper morphism $f : (Y, B_Y) \rightarrow X$ from an *embedded normal crossing pair* satisfying the following properties:

- (1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.
- (2) The natural map $\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil)$ induces an isomorphism

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor),$$

where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$.

- (3) The collection of subvarieties $\{C\}$ coincides with the image of (Y, B_Y) -strata that are not included in $X_{-\infty}$.

In Definition 3.2, we assume that (Y, B_Y) is a *globally embedded simple normal crossing pair*. On the other hand, in Definition 8.2, we only assume that (Y, B_Y) is an *embedded normal crossing pair*.

Remark 8.3. As was pointed out in Remark 3.4, a quasi-log scheme in Definition 8.2 was called a quasi-log *variety* in [A].

Remark 8.4. In [A], Ambro required that (Y, B_Y) is *embedded* for technical reasons and expected that this *extra* assumption is not necessary (see [A, Introduction]). On the other hand, the author thinks that the existence of the ambient space M of (Y, B_Y) makes the theory of quasi-log schemes more flexible and more powerful. Note that a key point of the main theorem in [F7] is to construct good ambient spaces for quasi-projective semi log canonical pairs after some suitable birational modifications.

Lemma 8.5 is essentially the same as Ambro's *embedded log transformations* in [A].

Lemma 8.5. *Let (Y, B_Y) be an embedded normal crossing pair and let M be the ambient space of (Y, B_Y) . Then there are a projective surjective morphism $\sigma : M' \rightarrow M$ from a smooth variety M' such that σ is a composition of blow-ups and a simple normal crossing pair (Z, B_Z) embedded into M' with the following properties.*

- (i) $\sigma : Z \rightarrow Y$ is surjective and $K_Z + B_Z = \sigma^*(K_Y + B_Y)$.
- (ii) $\sigma_* \mathcal{O}_Z([- (B_Z^{\leq 1})] - \lfloor B_Z^{\geq 1} \rfloor) \simeq \mathcal{O}_Y([- (B_Y^{\leq 1})] - \lfloor B_Y^{\geq 1} \rfloor)$.
- (iii) Let C' be a stratum of (Z, B_Z) . Then $\sigma(C')$ is a stratum of (Y, B_Y) or is contained in $\text{Supp } B_Y^{\geq 1}$. Let C be a stratum of (Y, B_Y) . Then there is a stratum C' of (Z, B_Z) such that $\sigma(C') = C$.

Proof. First, we can construct a sequence of blow-ups $M_k \rightarrow M_{k-1} \rightarrow \cdots \rightarrow M_0 = M$ with the following properties.

- (a) $\sigma_{i+1} : M_{i+1} \rightarrow M_i$ is the blow-up along a smooth stratum of Y_i for every i .
- (b) We put $Y_0 = Y$, $B_{Y_0} = B_Y$, and $Y_{i+1} = \sigma_{i+1}^{-1}(Y_i)$ with the reduced scheme structure.
- (c) Y_k is a simple normal crossing divisor on M_k .

We can check that $K_{Y_{i+1}} = \sigma_{i+1}^* K_{Y_i}$ for every i by the construction. We can directly check that

$$R^1 \sigma_{i+1*} \mathcal{O}_{M_{i+1}}(-Y_{i+1}) = 0$$

and

$$\sigma_{i+1*} \mathcal{O}_{M_{i+1}}(-Y_{i+1}) \simeq \mathcal{O}_{M_i}(-Y_i)$$

for every i . Therefore, by the diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{M_i}(-Y_i) & \longrightarrow & \mathcal{O}_{M_i} & \longrightarrow & \mathcal{O}_{Y_i} \longrightarrow 0 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \\ 0 & \longrightarrow & \sigma_{i+1*}\mathcal{O}_{M_{i+1}}(-Y_{i+1}) & \longrightarrow & \sigma_{i+1*}\mathcal{O}_{M_{i+1}} & \longrightarrow & \sigma_{i+1*}\mathcal{O}_{Y_{i+1}} \longrightarrow 0, \end{array}$$

we obtain $\sigma_{i+1*}\mathcal{O}_{Y_{i+1}} \simeq \mathcal{O}_{Y_i}$ for every i . We put $B_{Y_{i+1}} = \sigma_{i+1}^*B_{Y_i}$ for every i . Then, by replacing (Y, B_Y) and M with (Y_k, B_{Y_k}) and M_k , we may assume that Y is a simple normal crossing divisor on M .

Next, we can construct a sequence of blow-ups $M_k \rightarrow M_{k-1} \rightarrow \cdots \rightarrow M_0 = M$ with the following properties.

- (1) $\sigma_{i+1} : M_{i+1} \rightarrow M_i$ is the blow-up along a smooth stratum of (Y_i, B_{Y_i}) contained in $\text{Supp } B_{Y_i}$ for every i .
- (2) We put $Y_0 = Y$ and $B_{Y_0} = B_Y$. Let Y_{i+1} be the strict transform of Y_i on M_{i+1} for every i .
- (3) We put $K_{Y_{i+1}} + B_{Y_{i+1}} = \sigma_{i+1}^*(K_{Y_i} + B_{Y_i})$ for every i .
- (4) $\text{Supp } B_{Y_k}$ is a simple normal crossing divisor on Y_k .

Finally, by the construction, we can check the properties (i), (ii), and (iii) for $\sigma : M_k \rightarrow M$ and (Y_k, B_{Y_k}) by an easy calculation of discrepancy coefficients similar to the proof of Proposition 4.1. \square

Proposition 8.6. *Assume that (Y, B_Y) is an embedded simple normal crossing pair in Definition 8.2. Let M be the ambient space of (Y, B_Y) . Then, by taking some sequence of blow-ups of M , we may further assume that (Y, B_Y) is a globally embedded simple normal crossing pair in Definition 8.2.*

Proof. We can construct a sequence of blow-ups $M_k \rightarrow M_{k-1} \rightarrow \cdots \rightarrow M_0 = M$ with the following properties.

- (i) $\sigma_{i+1} : M_{i+1} \rightarrow M_i$ is the blow-up along a smooth irreducible component of $\text{Supp } B_{Y_i}$ for every $i \geq 0$.
- (ii) We put $Y_0 = Y$ and $B_{Y_0} = B_Y$. Let Y_{i+1} be the strict transform of Y_i for every $i \geq 0$.
- (iii) We define $K_{Y_{i+1}} + B_{Y_{i+1}} = \sigma_{i+1}^*(K_{Y_i} + B_{Y_i})$ for every $i \geq 0$.
- (iv) There exists an \mathbb{R} -divisor D on M_k such that $\text{Supp}(Y_k + D)$ is a simple normal crossing divisor on M_k and that $D|_{Y_k} = B_{Y_k}$.
- (v) $\sigma_*\mathcal{O}_{Y_k}(\lceil -(B_{Y_k}^{\leq 1}) \rceil - \lfloor B_{Y_k}^{> 1} \rfloor) \simeq \mathcal{O}_Y(\lceil -(B_Y^{\leq 1}) \rceil - \lfloor B_Y^{> 1} \rfloor)$, where $\sigma : M_k \rightarrow M_{k-1} \rightarrow \cdots \rightarrow M_0 = M$.

Moreover, we have:

- (vi) Let C' be a stratum of (Y_k, B_{Y_k}) . Then $\sigma(C')$ is a stratum of (Y, B_Y) . Let C be a stratum of (Y, B_Y) . Then there is a stratum C' of (Y_k, B_{Y_k}) such that $\sigma(C') = C$.

We note that we can directly check

$$\sigma_{i+1*}\mathcal{O}_{Y_{i+1}}(\lceil -(B_{Y_{i+1}}^{<1}) \rceil - \lfloor B_{Y_{i+1}}^{>1} \rfloor) \simeq \mathcal{O}_{Y_i}(\lceil -(B_{Y_i}^{<1}) \rceil - \lfloor B_{Y_i}^{>1} \rfloor)$$

for every $i \geq 0$ and the property (vi) by an easy calculation of discrepancy coefficients similar to the proof of Proposition 4.1. Then we replace M and (Y, B_Y) with M_k and (Y_k, B_{Y_k}) . \square

Therefore, by Lemma 8.5 and Proposition 8.6, Definition 3.2 is equivalent to Ambro's original definition of quasi-log schemes: Definition 8.2.

Theorem 8.7. *Definition 3.2 is equivalent to Definition 8.2.*

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